Remarks on the 3D Lattice Sieve

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Outline

Number Field Sieve

NFS Number Field Sieve

- Subexponential method for factoring and finding discrete logarithms.

NFS Main principle

- Find enough equations to construct a specific linear system.
- Find enough smooth algebraic numbers in specific number fields.
- Smooth numbers are identified via sieving a predefined space (lattice).

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Algebraic factor base

- Different views in NFS: algebraic numbers, ideals vs. points in space, memory cells.
- Factor base: A set of ideals with the norm $< B$.
- Prime ideal — a set of algebraic numbers:
  - $\mathfrak{p} = 7\mathbb{Z}[\alpha] + (3+\alpha)\mathbb{Z}[\alpha]$
  - a lattice

$$\begin{pmatrix} 7 & 0 \\ 3 & 1 \end{pmatrix}$$

  - every number has norm divisible by 7, main ideal divisible by $\mathfrak{p}$

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Sieving

- Sieving region:
  - A set of points representing algebraic integers in 2 (or more) number fields.
  - Rectangle, lattice, box, ...
  - Memory cells with precomputed (approximations of) norms...
- Sieving:
  - For each prime ideal in the factor base, mark corresponding points in the sieve region.
  - Check which points are "marked enough"
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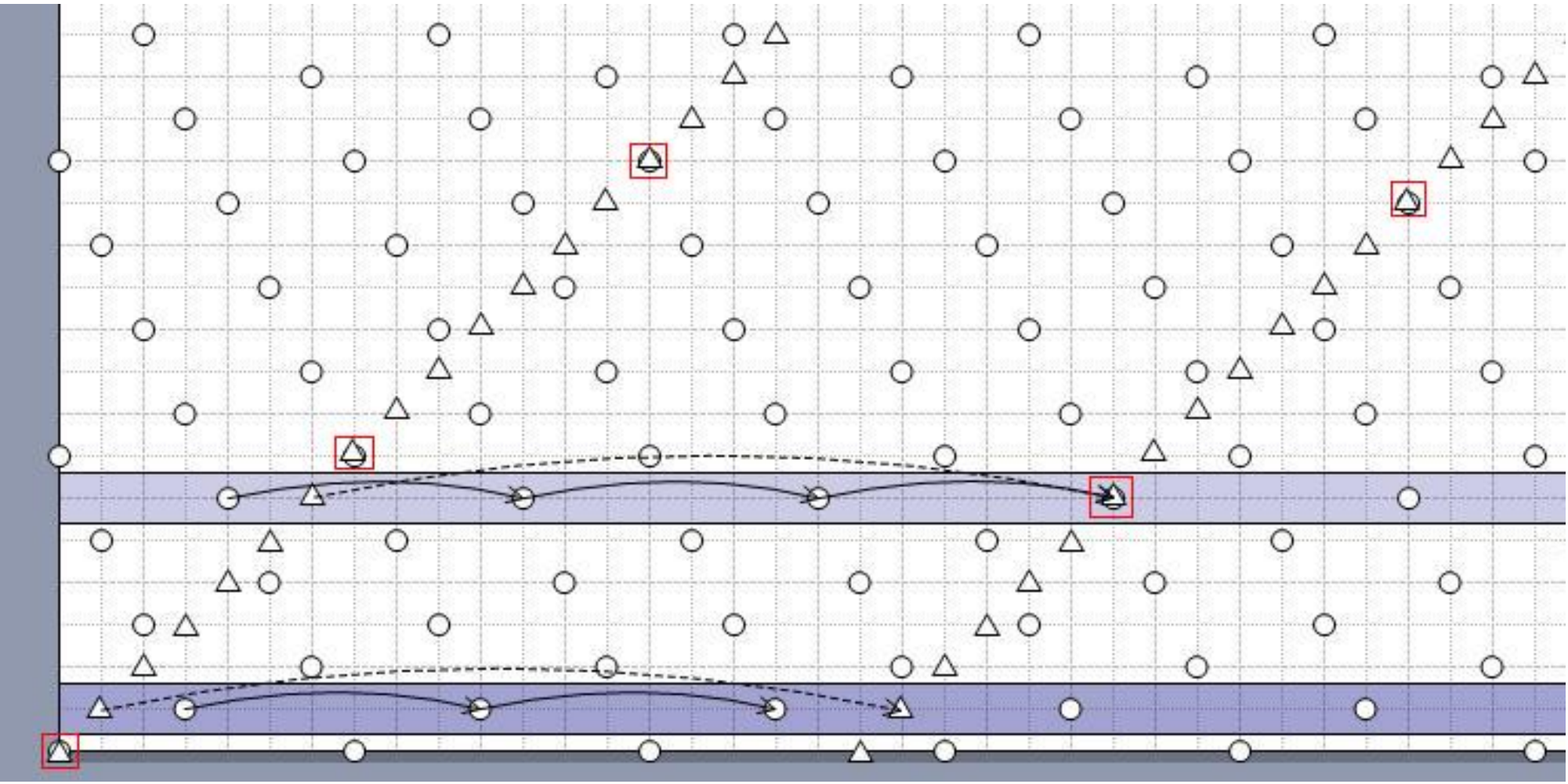
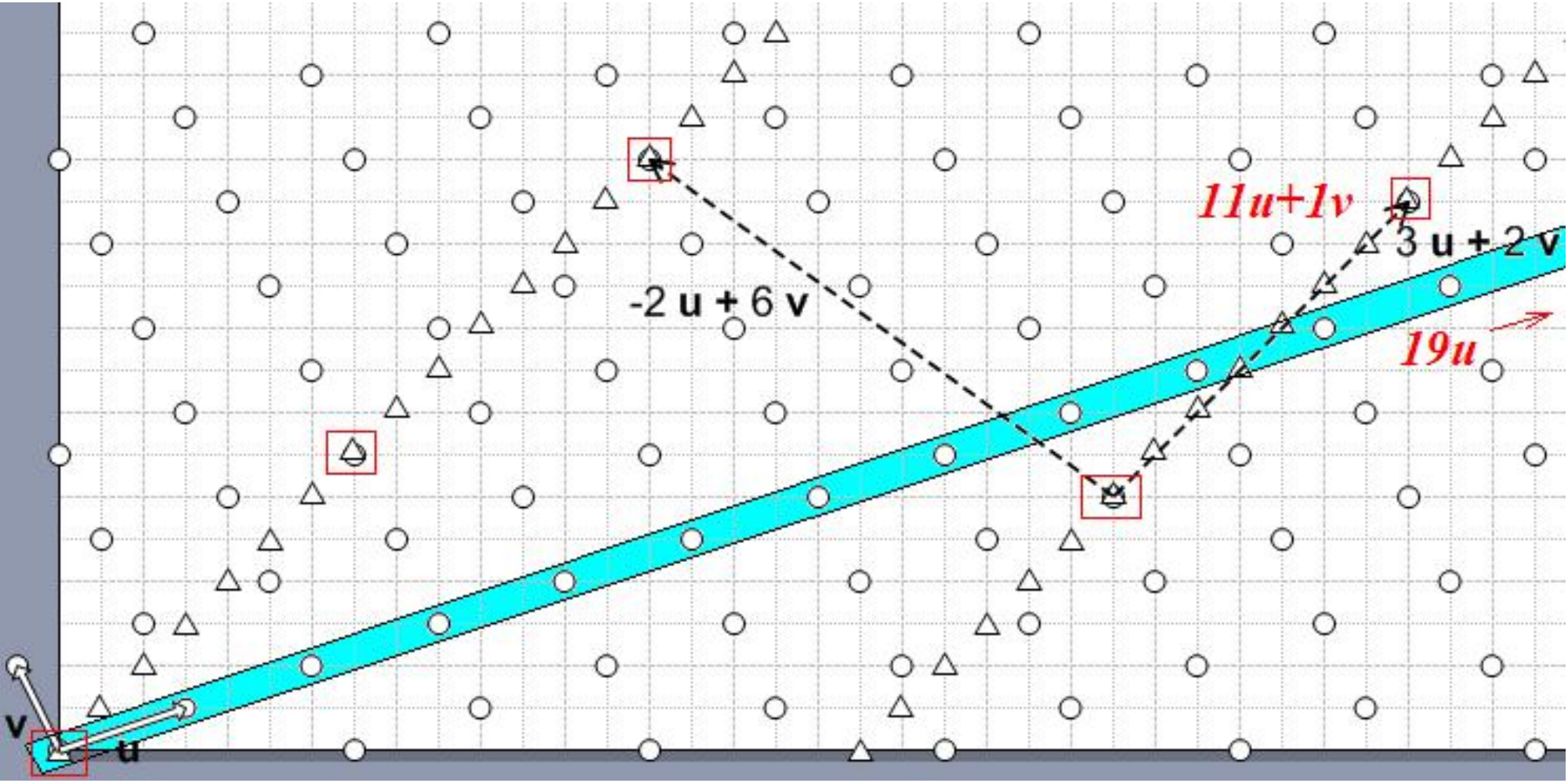
Line sieve

- Main principle: Sieve the region along one-dimensional lines.
- Additional optimizations can be used.



Special-q sieve

- Also: Lattice sieve.
- Main principle: Take a sub-lattice corresponding to a chosen ideal (special-$\mathfrak{q}$) and sieve this smaller region.
- If the region is large enough, can combine with line sieve.
- If the region is small to fit in memory, points of ideals can be enumerated.


11u+1v
3 u + 2 v
-2 u + 6 v
19u
v
u

Lattice sieve

- Main advantage — smaller number of points to examine with a guaranteed factor, smaller work-factor.
- Disadvantages:
  - Time to recompute bases of factor base ideals.
  - Faster growth of the norms (strong limit on sieve bounds).
  - Collisions (requires post-filtering).

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3D sieve

- Required for the NFS for DLP in degree 6 fields.
- Extend the sieve region to a box, recompute 3D bases of ideals, e.g.

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- Parameters:
  - Region size: $2^{19} \times 2^{13} \times 2^{10}$
  - Factor base size: 893707 ideals ($B = 6500000$)
  - Number of equations: 1077984
  - Total time: 2 087 070 s
- Only 89987 equations with more than 2 factors above 400000-th ideal, ordered by frequency of the occurence in smooth equations.

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Line sieving with large factors

- After sieve reduction to first 400000 ideals ($B' = 2757229$):
  - B'-smooth equations: 119053 (11%)
  - B-smooth equations (single large factor): 306198 (28%)
  - Total time: 232358 s (11%, but faster computers...)
- Possible problem with detection:
  - In general: Higher tolerance = longer total time (due to factoring).
  - Different behaviour with large factors.

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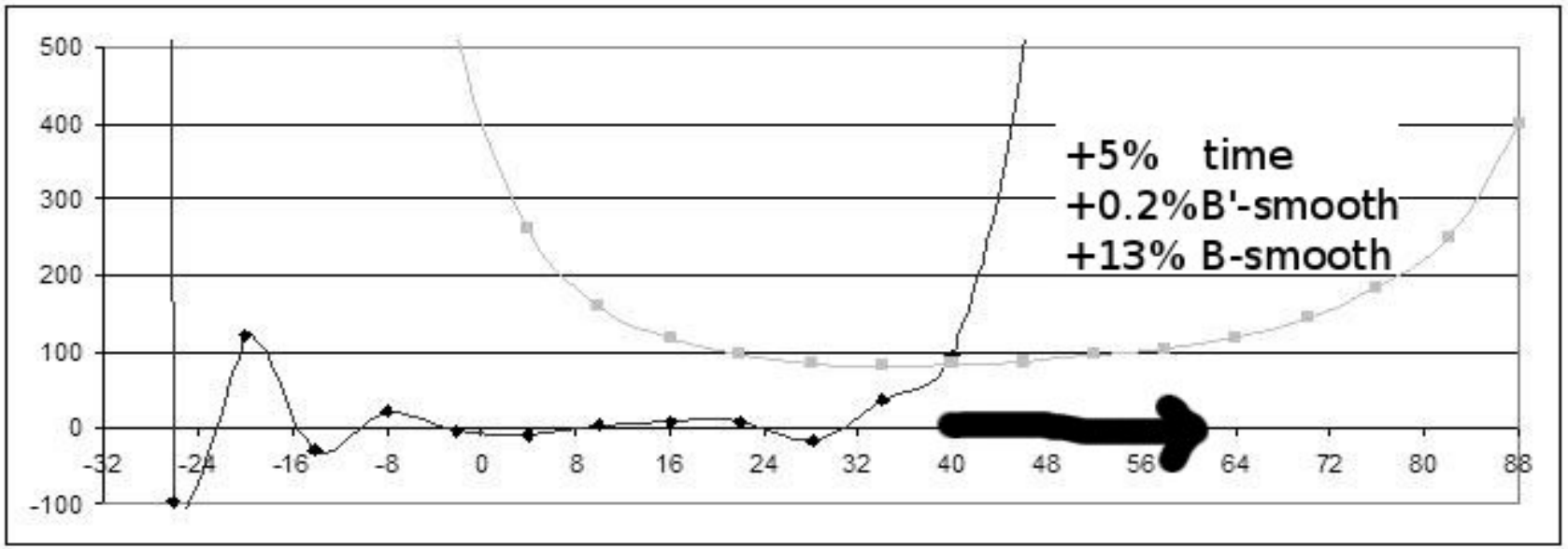

Figure: Additional time to find one smooth equation and average sieving time per smooth equation with a given sieve tolerance.

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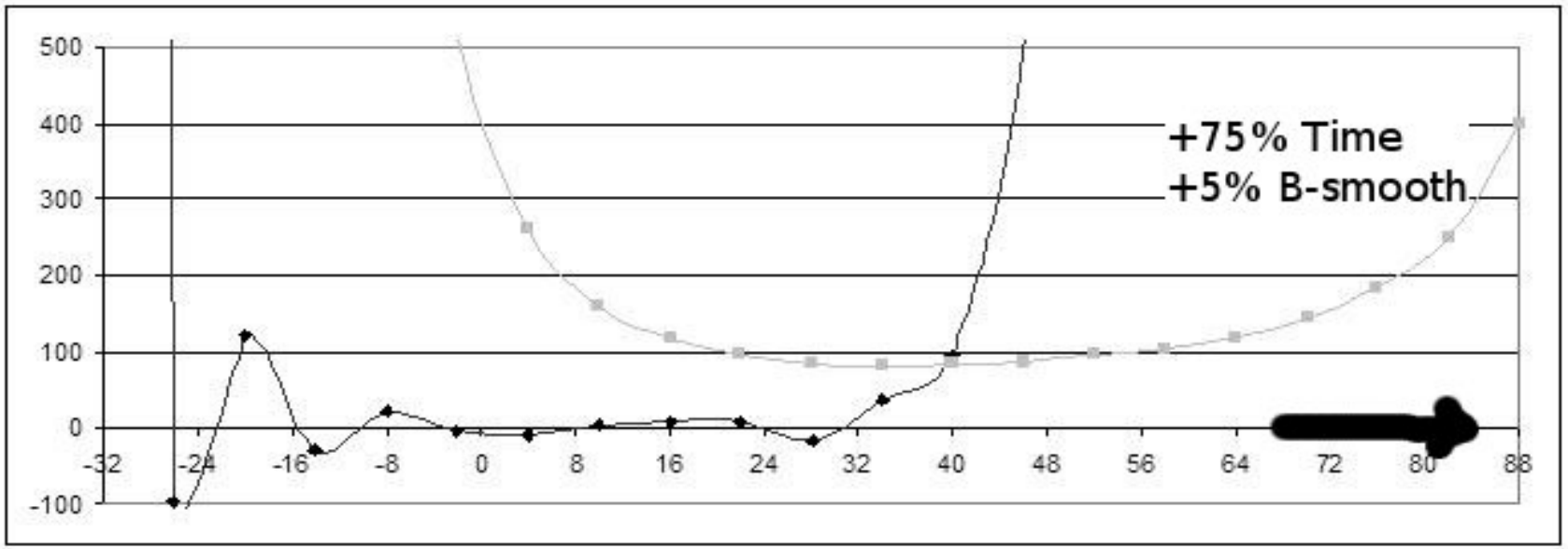

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Lattice sieving — large primes

- Special q-sieve, with large primes ($B' < q < B$), statistics per 20000 special-q's:
  - Equations per special prime: .90 $\rightarrow$ .63 (usable: .35 $\rightarrow$ .19)
  - Much slower due to base transformation (2-times longer than plain sieving, can be fine-tuned)
  - Not enough equations... (approx. 100000 new B'-smooth eqs., 50% missing)
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  - 0.02% of the whole base
  - covers 80 % of all equations
- Experiment with reduced sieve region:
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  - possible speedup using partial base large factors,
  - if we can identify a-priori primes that appear frequently...
- Lattice sieving with large primes is ineffective:
  - too many base transformations
  - sieve region too slow for effective implementation
- Lattice sieving with small primes still too slow:
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- Possible speedup: More effective factoring of sets of semi-smooth numbers with a partial sieve.

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